LAST TIME: Cross Product 9/3/21

ex. $\vec{u} = \langle 5, 3, -1 \rangle$, $\vec{v} = \langle 8, 4, 2 \rangle$ *show all steps!

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & 3 & -1 \\ 8 & 4 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & -1 \\ 4 & 2 \end{vmatrix} \vec{i} - \begin{vmatrix} 5 & -1 \\ 8 & 2 \end{vmatrix} \vec{j} + \begin{vmatrix} 5 & 3 \\ 8 & 4 \end{vmatrix} \vec{k}$$

$$= (3 \cdot 2 - (-1) \cdot 4)\vec{i} - (5 \cdot 2 - (-1) \cdot 8)\vec{j} + (5 \cdot 4 - 3 \cdot 8)\vec{k}$$

$$= (6 + 4)i - (10 + 8)\vec{j} + (20 - 24)\vec{k} = 10i - 18j - 4\vec{k} = \boxed{\langle 10, -18, -4 \rangle}$$

↳ Recall: prop (Algebraic Properties of Cross product)

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ and $c \in \mathbb{R}$

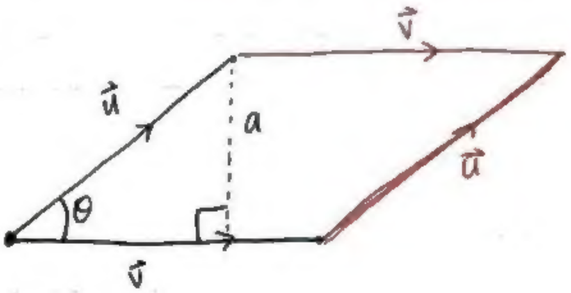
1. $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
2. $(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v}) = \vec{u} \times (c\vec{v})$
3. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
4. $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$
5. $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$
6. $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})(\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$
7. $\vec{u}$ and $\vec{v}$ are both orthogonal to $\vec{u} \times \vec{v}$
8. $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}|\sin\theta$ with θ the angle between $\vec{u}, \vec{v}$
9. $\vec{u} \times \vec{v} = 0$ iff $\vec{u}$ and $\vec{v}$ are parallel.

(7–9:) geometric properties

ex. Take $\vec{v} \times \vec{u}$ for the given vectors.

$$\vec{v} \times \vec{u} = -\vec{u} \times \vec{v}$$
$$= -\langle 10, -18, -4 \rangle$$
$$= \langle -10, 18, 4 \rangle$$

↳ $\vec{u} \times \vec{v}$ is computed w "right hand rule"



$\sin(\theta) = \dfrac{a}{|\vec{v}|}$ ∴ $a = |\vec{v}|\sin(\theta)$

∴ area of parallelogram determined by $\vec{u}$ & $\vec{v}$ is

$A = (\text{base})(\text{height}) = |\vec{u}|a = |\vec{u}||\vec{v}|\sin\theta$

Proof of prop ⑧: We use the algebraic properties to compute $|\vec{u}\times\vec{v}|^2$

$= |\vec{u}\times\vec{v}|^2 = (\vec{u}\times\vec{v})\cdot(\vec{u}\times\vec{v})$ (properties of dot product)

$= \vec{u}\cdot(\vec{v}\times(\vec{u}\times\vec{v}))$ (part ⑤ of properties of cross)

$= \vec{u}\cdot((\vec{v}\cdot\vec{v})\vec{u}-(\vec{v}\cdot\vec{u})\vec{v})$ (part ⑥ of properties of cross)

$= \vec{u}\cdot(\vec{v}\cdot\vec{v})\vec{u}-\vec{u}((\vec{v}\cdot\vec{u})\vec{v})$ (properties of dot)

$= (\vec{v}\cdot\vec{v})(\vec{u}\cdot\vec{u})-(\vec{v}\cdot\vec{u})(\vec{u}\cdot\vec{v})$ (properties of dot)

$= |\vec{v}|^2|\vec{u}|^2-(\vec{u}\cdot\vec{v})^2$ (properties of dot)

$= (|\vec{v}||\vec{u}|)^2-(|\vec{u}||\vec{v}|\cos(\theta))^2$ ← geometric interpretation of dot product

$= (|\vec{u}||\vec{v}|)^2-(|\vec{u}||\vec{v}|)^2\cos^2(\theta)$

$= (|\vec{u}||\vec{v}|)^2(1-\cos^2(\theta))$

$= (|\vec{u}||\vec{v}|)^2\sin^2(\theta)$

$= (|\vec{u}||\vec{v}|\sin(\theta))^2$

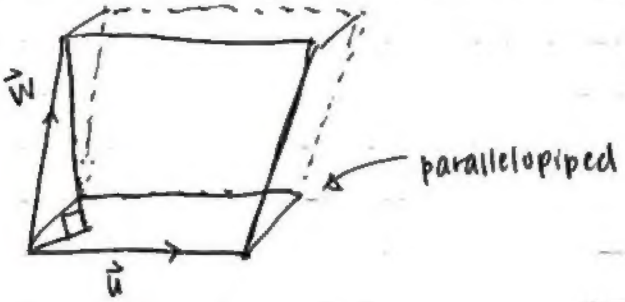
$\therefore |\vec{u}\times\vec{v}|^2 = (|\vec{u}||\vec{v}|\sin(\theta))^2$

↳ Recall: θ is the angle between vectors $\vec{u}$ and $\vec{v}$, so $\theta \in [0,\pi]$, so $\sin(\theta)\geq 0$

$\boxed{\text{Hence } |\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin(\theta)}$

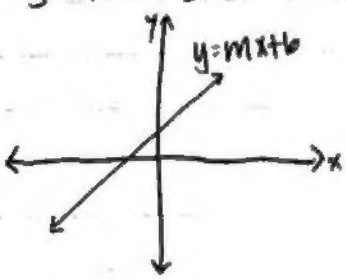
Cor: The magnitude $|\vec{u}\times\vec{v}|$ is the area of the parallelogram determined by $\vec{u}$ & $\vec{v}$

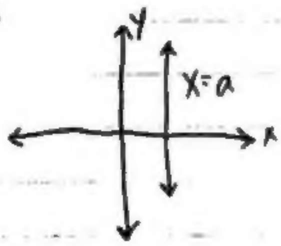
prop: The scalar triple product $\vec{u}\cdot(\vec{v}\times\vec{w})$ is the signed volume of the parallelepiped by $\vec{u}$, $\vec{v}$, & $\vec{w}$



↳ Proof is on website

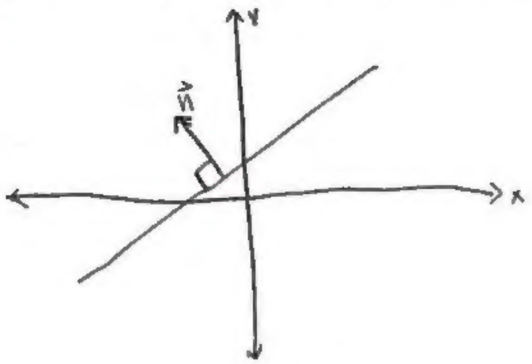
§ 12.5: Lines & Planes





$ax+by-c=0$

In 2-space:



line is the set of points w/ $\vec{n} \cdot \vec{x} = c$

- Generalize this equation to 3-space.

$\vec{n} \cdot \vec{x} = d$

i.e. $\langle a, b, c \rangle \cdot \langle x, y, z \rangle = d$

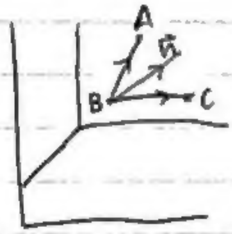
i.e. $ax + by + cz = d$

↳ This is a plane in 3-space

NB: If we know two nonparallel vectors $\vec{u}$ & $\vec{v}$ which lie in the plane (i.e. their head & tail can be expressed on the plane at the same time) then $\vec{n} = \vec{u} \times \vec{v}$ is a normal vector to the plane. i.e. it's perpendicular to every vector in the plane

ex: Find the vector equation of the plane through the points $(0,1,3)$, $(4,9,7)$, and $(1,2,3)$

solution:



Note that the vectors $\vec{u} = \langle 4-0, 9-1, 7-3 \rangle = \langle 4, 8, 4 \rangle$

and $\vec{v} = \langle 1-0, 2-1, 3-3 \rangle = \langle 1, 1, 0 \rangle$

lie in the desired plane

∴ we may use normal vector $\vec{n} = \vec{u} \times \vec{v}$

$$\therefore \vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 8 & 4 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= \langle -4, -(-4), -4 \rangle = \langle 1, -1, 1 \rangle$$

∴ the plane has equation

$\vec{n} \cdot \vec{x} = d$ for some constant d

i.e. $\langle 1, -1, 1 \rangle \cdot \langle x, y, z \rangle = d$

i.e. $x - y + z = d$

↳ To compute d, use $(0,1,3)$

$d = 0 - 1 + 3 = 2$, so the plane has equation $x - y + z = 2$